Approximate Analysis of Free-Mixing Flows

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I. Introduction

AS outgrowths of the recent interest in the wakes of hypersonic re-entry vehicles, detailed numerical solutions of the axisymmetric and two-dimensional free-mixing equations have been developed.[1, 2] Although such methods are necessary to predict flow properties in chemically reacting nonequilibrium flows, their use for less complex flow problems may be prohibited by high computer expense; it is then desirable to have available means to develop accurate approximate solutions.

Two commonly employed methods to generate approximate solutions to the boundary-layer equations are the integral method[3] and linearization.[4, 5] The integral method provides rapid solutions with minimum effort; however, the form of the assumed profiles generally puts undesirable constraints on the applicability of the results. Solutions of the linearized free-mixing equations can be generated for arbitrary initial conditions; these solutions demonstrate the proper "downstream" behavior, but are of limited use in the "upstream" regions.

The purpose of the present work is to elaborate on a technique advanced by Libby and Schetz[6] for the "improvement" of a linearized solution. The method is based upon a stretching of the coordinates, the stretching factors being found by forcing the linearized solution to satisfy certain aspects of the nonlinear problem.

As an extension of ideas put forth by Carrier[5] and Lewis and Carrier,[7] Libby and Schetz[6] solved the two-dimensional laminar boundary-layer problem of slot injection into a constant pressure moving stream by first determining a linearized solution, then replacing the streamwise coordinate x by a function $\xi(x)$, and then forcing the linearized solution in terms of ξ to satisfy the "exact" momentum integral equation. Since $d/dx = (d\xi/dx)(d/d\xi)$, the operations just described yielded a simple differential equation for the stretched coordinate ξ. The method proved to be successful for the problem originally treated, and has since been applied to other situations.[8] Two-dimensional symmetric jets (in a moving ambient stream) were analyzed with streamwise coordinate stretching being accomplished by forcing the linearized solution in terms of ξ to satisfy the exact equation written on the symmetry axis. Another approach used by Schetz[9] is pointwise collocation of a linearized solution. In this case, parameters are introduced into the argument of the linearized solution; these parameters are evaluated by forcing the resulting expression to satisfy the exact nonlinear differential equation at discrete points.

The present note describes a scheme, applicable to free-mixing flows, according to which the forenoted ideas are combined; the momentum integral equation is used to stretch the radial coordinate, whereas the differential equation, written on an appropriate line, is used to stretch the streamwise coordinate. A comparison of the results of an illustrative calculation with an exact numerical solution[1] shows excellent agreement.

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A related investigation is Hsia's study of the flow in the inlet region of a two-dimensional channel.[12] There, a linearized solution was obtained by replacing the convective operator, $u\, \partial u/\partial x + v\, \partial u/\partial y$, with $k_1(x)\, \partial u/\partial x + k_2(x)$. The functions $k_1(x)$ and $k_2(x)$ were then determined by forcing the linearized solution to satisfy both the momentum integral and the exact momentum equation for the channel centerline.

II. Analysis

For the present, incompressible flow is considered; a latter section notes the necessary assumptions in order to use the results for compressible flow. Also, constant pressure flow is assumed; however, in the case of incompressible flow, a streamwise pressure gradient is easily accommodated.

The streamwise momentum equation for constant pressure flow is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\nu}{y^j}\frac{\partial}{\partial y}\left(y^j\frac{\partial u}{\partial y}\right) \tag{1}$$

in which $j = 0, 1$ for two-dimensional and axisymmetric flows, respectively, and x, u and y, v are streamwise and radial coordinates and associated velocities, respectively. Boundary and initial conditions on u are

$$u(x, \pm\infty) = u_e \qquad u(o,y) = u_i(y) \tag{2}$$

For turbulent flow, Eq. (1) is assumed to apply to the temporal mean flow properties, and ν is taken to be the eddy diffusivity (which is at most a function of x), rather than the kinematic viscosity.

Introducing

$$\bar{u} = \frac{u}{u_e} \qquad \bar{y} = \frac{y}{L} \qquad \bar{x} = \int_0^x \frac{\nu}{u_e L^2}\,dx'$$

(with L being a reference length), and applying the Oseen linearization (5) to Eq. (1), there results

$$\frac{\partial \Delta u}{\partial \bar{x}} = \frac{1}{\bar{y}^j}\frac{\partial}{\partial \bar{y}}\left[\bar{y}^j\frac{\partial \Delta \bar{u}}{\partial y}\right] \tag{3}$$

where $\bar{u} = 1 \pm \Delta\bar{u}$ (positive sign for jet-like flows and negative sign for wake-like flows).

The solution of Eq. (3), subject to arbitrary initial conditions, may be determined by standard techniques (e.g., Ref. 10). The "improvement" of the linearized result is obtained by introducing stretched coordinates $\xi_1(\bar{x})$ and $\xi_2(\bar{x},\bar{y}) \equiv \bar{y}f(\xi_1)$, and interpreting the linearized solution as $\Delta u(\xi_1,\xi_2)$; the functions $\xi_1(x)$ and $f(\xi_1)$ are determined by requiring that $\Delta u(\xi_1,\xi_2)$ satisfy the momentum integral equation

$$\frac{d}{dx}\int_A \Delta\bar{u}(1 - \Delta\bar{u})\,dA = 0 \tag{4a}$$

and the partial differential equation written on the axis $\bar{y} = 0$

$$\left\{(1 - \Delta\bar{u})\frac{\partial\Delta\bar{u}}{\partial\bar{x}} = \frac{1}{\bar{y}^j}\frac{\partial}{\partial\bar{y}}\left[\bar{y}^j\frac{\partial\Delta\bar{u}}{\partial\bar{y}}\right]\right\}_{\bar{y}=0} \tag{5a}$$

Excluding asymmetric two-dimensional flows, Eqs. (4a) and (5a) may be written explicitly as

$$\frac{d}{d\xi_1}\left\{f^{-(1+j)}\int_0^\infty \Delta\bar{u}(1 - \Delta\bar{u})\xi_2{}^j d\xi_2\right\} = 0 \tag{4b}$$

$$\left\{(1 - \Delta\bar{u})\frac{\partial\Delta\bar{u}}{\partial\xi_1}\frac{d\xi_1}{d\bar{x}} = (1 + j)f^2\frac{\partial^2\Delta\bar{u}}{\partial\xi_2{}^2}\right\}_{\xi_2=0} \tag{5b}$$

Since $\Delta\bar{u}(\xi_1,\xi_2)$ is defined as a solution of the linearized equation, Eq. (5b) reduces to

$$d\xi_1/d\bar{x} = f^2/(1 - \Delta\bar{u}) \tag{5c}$$

Equation (5c) may be integrated once $f(\xi_1)$ is obtained from Eq. (4b); the lower limit of integration being $\bar{x} = \xi_1 = 0$ for Eq. (5c), and the initial condition for Eq. (4b) being $f = 1$ at $\xi_1 = 0$.

It is seen that the present analysis is vaguely similar to the standard integral method for free mixing in the sense that the momentum integral and the axis differential equation are used to determine the two functions ξ_1 and ξ_2 in a profile generated by a linearized solution. In the integral method, the same two conditions would be used to determine the parameters of the assumed profile, namely, the axis velocity and the width of the mixing layer.

III. Compressibility Effects

For the present purposes it is convenient to consider the compressible flow momentum equation in the Von Mises plane, viz.,

$$\frac{\partial u}{\partial s} = \frac{1}{\psi^j}\frac{\partial}{\partial \psi}\left[\rho\mu\left(\frac{y}{n}\right)^{2j}\frac{n^{2j}u}{\psi^j}\frac{\partial u}{\partial \psi}\right] \tag{6}$$

where

$$s = x \qquad \rho u y^j = \psi^j \frac{\partial \psi}{\partial y} \qquad -\rho v y^j = \psi^j \frac{\partial \psi}{\partial x}$$

$$n^{1+j} = (1+j)\int_0^y \frac{\rho\, y^j\, dy}{\rho_e}$$

Equation (6) assumes incompressible form if $C = \rho\mu(y/n)^{2j}$ is a function of x or a constant.

The case of turbulent flow has been investigated by Ting and Libby,[11] who used the forenoted expression to define a turbulent diffusivity for compressible flow, i.e.,

$$[\rho\mu(y/n)^{2j}]_{\text{comp}} \equiv \rho_e\mu_{\text{incomp}}$$

For laminar flow, two interpretations of the preceding are possible: first, if the more common assumption, $\rho\mu$ = function of x, is made, then it is necessary to assume that y/n and therefore ρ/ρ_e are functions of x. Viewed as ρ/ρ_e independent of y/δ (δ being the local width of the mixing zone), such an approximation may be satisfactory for some flows. The second interpretation is that the combined term eliminates y dependence. In this case, with T being the temperature, $\rho \sim 1/T$, $\mu \sim T^N$, and for modest temperature variations $y/n \sim T^{1/1+j}$ (roughly). Then

$$C = \rho\mu(y/n)^{2j} \sim T^{N-[(1-j/1+j)]}$$

The only conclusive result obtained here is the familiar one, viz., that with $N = 1$, C is a constant for two-dimensional flow. It is thus seen that the "transformation" of the momentum equation is imprecise and may only be useful in a qualitative sense.

Accepting one of the preceding conditions, once the velocity is available the energy equation and, for mixtures of gases, the species continuity equations may be solved since these equations are linear.

IV. Illustrative Case

The method just described is illustrated here for a flow having an initial velocity profile given by

$$\Delta\bar{u}(0,\bar{y}) = Ae^{-B\bar{y}^2} \tag{7}$$

The linearized solution yields

$$\Delta\bar{u} = \frac{A}{1+4\xi_1}\exp\left(-\frac{B\xi_2{}^2}{1+4\xi_1}\right) \tag{8}$$

Substituting this result into Eqs. (4b) and (5b),

$$f^2 = \frac{2A}{2-A}\left[\frac{1}{A} - \frac{1}{2(1+4\xi_1)}\right] \tag{9a}$$

$$\bar{x} = \left(\frac{2-A}{2B}\right)\left[\xi_1 - \frac{A}{8}\ln\left(\frac{8\xi_1+2-A}{2-A}\right)\right] \tag{9b}$$

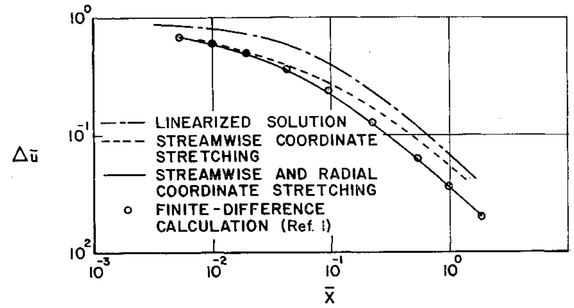

Fig. 1 Streamwise variation of axis velocity defect.

Alternately, if only the streamwise coordinate is stretched, there results

$$f \equiv 1 \tag{10a}$$

$$\bar{x} = \xi_1 - (A/4)\ln(1+4\xi_1) \tag{10b}$$

Numerical results are presented in Figs. 1 and 2 for a wakelike flow having $A = 0.9$ and $B = 3.1$ (and the square of the reference length L^2 being equal to the "drag area" C_DA). The figures show the streamwise variation of the wake axis velocity defect and velocity profiles at two downstream locations. The results are shown for the linearized solution, the solution with only the streamwise coordinate stretching [Eqs. (10a) and (10b)], and the solution with both streamwise and radial coordinate stretching [Eqs. (9a) and (9b)]. Also indicated are results of an exact numerical solution obtained by the method described in Ref. 1.

It is seen that excellent agreement with the exact solution is obtained when both coordinates are stretched. As anticipated for this case, the linearized solution is poor since the linearization is valid only if $\Delta u(0,y) \ll 1$.

A word of caution should be noted here with regard to the use of the method for arbitrary initial conditions. In their investigation of slot injection along an adiabatic plate, Libby and Schetz[6] found that the coordinate stretching in both the momentum and energy equations was possible only if the ratio of jet to free-stream velocity was in the range $\frac{1}{3}$ to 1.0. It is conceivable that similar limitations may exist in certain free-mixing flows.‡ However, due to the wide variety of initial conditions of interest, it is not possible to identify these cases here.

As a final comment, it is noted that applications of the present analysis to some types of three-dimensional steady and unsteady wake flows have been made; results are reported in Ref. 13.

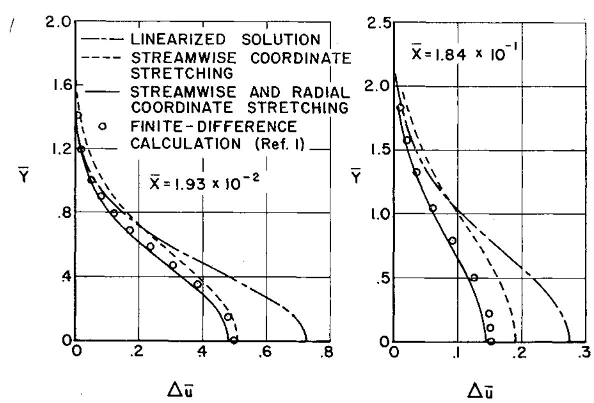

Fig. 2 Velocity profiles.

‡ For example, the results of the illustrative case are only meaningful if Eqs. (9a) and (9b) yield $f > 0$ and $\bar{x} > 0$. Moreover, the jet-like flow with $A = 2$ does not allow both radial and streamwise coördinate stretching [Eq. (9)].

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Ignitability of Nonhypergolic Propellants in Presence of Potassium Permanganate

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Many nonhypergolic fuels can be made hypergolic by introducing suitable additives.[1] In some cases, introduction of a hot surface brings about ignition, which presumably makes use of the fact that the temperature is increased in the zone of reaction. Since ignition is preceded by oxidative degradation processes, it seems that these would be accelerated by the use of stronger oxidizing agents. In this manner, many nonhypergolic fuels can be made hypergolic. The purpose of this note is to report the result of investigations undertaken from this angle.

An increasing amount of potassium permanganate was added to red fuming nitric acid, and the ignitability of various alcohols was tested with it. It was found that methyl alcohol, ethyl alcohol, propyl alcohol, isopropyl alcohol, butyl alcohol, secondary butyl alcohol and tertiary butyl alcohol all become hypergolic when 20% potassium permanganate is used. The ignition delay is below 0.3 sec in all cases. Studies were undertaken to elucidate the mechanism. The essential steps involved are the following:

$$\text{alcohol} \rightarrow \text{aldehyde or ketone} \rightarrow \text{acid} \rightarrow \text{degradation}$$

The intermediates in this reaction could be identified. As further confirmation of the mechanism, the ignitability of aldehydes and corresponding ketones was investigated. It was found that these ignite with red fuming nitric acid, which contains 10% potassium permanganate.

The role of potassium permanganate was investigated. It may be noted that only freshly dissolved potassium permanganate in red fuming nitric acid is effective. This gave us a strong suspicion that atomic oxygen is produced which acts as a much stronger oxidizing agent. This conclusion is supported by the fact that benzene also ignites with red fuming nitric acid containing potassium permanganate. Carbon disulfide also burns with a steel blue flame. However, the intriguing fact is that no reaction occurs with white fuming nitric acid. The role of NO_2 in the ignition reaction is not clear. Further studies are in progress.

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Normal Shock-Wave Properties in Imperfect Air and Nitrogen

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Because of the current interest in shock tube applications in the study of so-called real gas effects in gas dynamics, and since previous results did not include important imperfect gas‡ effects at high densities, recalculation and extension of previously published perfect gas calculations was necessary. The purpose of this note is to draw attention to the imperfect gas effects on the normal shock-wave properties especially at high shock strengths and quiescent gas pressures $p_1 \gtrsim 1$ atm.

Normal shock-wave properties have been computed for air[1] and nitrogen[2] in the range $M_S = 6(1)30$ into an ideal gas at a temperature of 300°K and pressures in the range from 10^{-4} to 10^3 cm Hg. The calculations were based on the recent thermodynamic data for imperfect air[3–5] and nitrogen.[3, 6, 7] Charts were presented[1, 2] for incident and reflected shock-wave conditions, stagnation conditions upstream and down-

Table 1 Imperfect and perfect gas normal shock-wave conditions in air

	Lewis and Burgess[1]	Feldman[8]
Air model	Imperfect	Perfect
M_S range	6–30	6–25
p_1 range (cm Hg)	10^{-4}–10^3	10^{-3}–76
Regions (see Fig. 2)	2, 2s, 20′, 20, 5	2, 2s, 20′, 5
Gasdynamic quantities	$p, \rho, T, h, u, a, Z, S$	p, ρ, T, h, u, Z^*

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‡ An ideal gas obeys $p = \rho RT$, $h = C_pT$, and $\gamma = C_p/C_v =$ const. A perfect gas will denote one obeying $p = Z^*\rho RT$ which includes dissociation and ionization neglecting intermolecular effects. An imperfect gas obeys $p = Z\rho RT$ which includes dissociation, ionization, and intermolecular forces. Local thermodynamic [i.e., thermal, mechanical (pressure), and chemical] equilibrium is assumed to exist for all conditions.